

# VECTOR-VALUED SPECTRA OF BANACH ALGEBRA VALUED CONTINUOUS FUNCTIONS

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**ABSTRACT.** Given a compact space  $X$ , a commutative Banach algebra  $A$ , and an  $A$ -valued function algebra  $\mathcal{A}$  on  $X$ , the notions of vector-valued spectrum of functions  $f \in \mathcal{A}$  are discussed. The  $A$ -valued spectrum  $\vec{\text{SP}}_A(f)$  of every  $f \in \mathcal{A}$  is defined in such a way that  $f(X) \subset \vec{\text{SP}}_A(f)$ . Utilizing the  $A$ -characters introduced in (M. Abtahi, *Vector-valued characters on vector-valued function algebras*, arXiv:1509.09215 [math.FA]), it is proved that  $\vec{\text{SP}}_A(f) = \{\Psi(f) : \Psi \text{ is an } A\text{-character of } \mathcal{A}\}$ . For the so-called natural  $A$ -valued function algebras, such as  $C(X, A)$  and  $\text{Lip}(X, A)$ , we see that  $\vec{\text{SP}}_A(f) = f(X)$ . When  $A = \mathbb{C}$ , Banach  $A$ -valued function algebras reduce to Banach function algebras,  $A$ -characters reduce to characters, and  $A$ -valued spectrums reduce to usual spectrums.

## 1. INTRODUCTION

Let  $A$  be a commutative complex Banach algebra with identity  $\mathbf{1}$ , and let  $\mathfrak{M}(A)$  denote the character space (maximal ideal space) of  $A$ , that is, the set of all nonzero multiplicative linear functionals  $\phi : A \rightarrow \mathbb{C}$ , endowed with the Gelfand topology. Given  $a \in A$ , the spectrum  $\text{SP}(a)$  of  $a$  consists of those complex numbers  $\lambda$  for which  $\lambda\mathbf{1} - a$  is not invertible in  $A$ . It is proved that  $\text{SP}(a)$  is a compact set and that

$$\text{SP}(a) = \{\phi(a) : \phi \in \mathfrak{M}(A)\}. \quad (1.1)$$

**1.1. Vector-valued Spectra.** Let  $n$  be a positive integer and  $\mathbf{a} = (a_1, \dots, a_n)$  be an  $n$ -tuples of elements of  $A$ . The *joint spectrum* of  $\mathbf{a}$ , again denoted by  $\text{SP}(\mathbf{a})$ , is defined to be the set of all  $n$ -tuples  $(\lambda_1, \dots, \lambda_n)$  of scalars in  $\mathbb{C}$  such that the identity element  $\mathbf{1}$  does not belong to the ideal generated by  $\{\lambda_i\mathbf{1} - a_i : 1 \leq i \leq n\}$ . That is,

$$\text{SP}(\mathbf{a}) = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \mathbf{1} \notin \sum_{i=1}^n A(\lambda_i\mathbf{1} - a_i) \right\}. \quad (1.2)$$

Similar to the case where  $n = 1$ , it is proved (e.g. [11]) that the joint spectrum  $\text{SP}(\mathbf{a})$  is a compact set in  $\mathbb{C}^n$  and that

$$\text{SP}(\mathbf{a}) = \{(\phi(a_1), \dots, \phi(a_n)) : \phi \in \mathfrak{M}(A)\}. \quad (1.3)$$

If one sets  $X = \{1, \dots, n\}$ , every  $n$ -tuples  $\mathbf{a} = (a_1, \dots, a_n)$  can be seen as an  $A$ -valued function  $\mathbf{a} : X \rightarrow A$ ,  $i \mapsto a_i$ . Similarly, every  $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  is a function from  $X$  into  $\mathbb{C}$ . For every  $\phi \in \mathfrak{M}(A)$ , the composition of  $\mathbf{a} : X \rightarrow A$  and  $\phi : A \rightarrow \mathbb{C}$  gives  $\phi \circ \mathbf{a} : X \rightarrow \mathbb{C}$ ,  $i \mapsto \phi(a_i)$ . With this convention, equality (1.3) is rephrased as follows:

$$\text{SP}(\mathbf{a}) = \{\phi \circ \mathbf{a} : \phi \in \mathfrak{M}(A)\}. \quad (1.4)$$

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In general, let  $X$  be a nonempty set, and let  $f : X \rightarrow A$  be an arbitrary function. The *vector-valued spectrum* of  $f$  is defined to be

$$\vec{\text{SP}}(f) = \left\{ \lambda : X \rightarrow \mathbb{C} : \mathbf{1} \notin \sum_{x \in F} A(\lambda(x)\mathbf{1} - f(x)) \right\}, \quad (1.5)$$

where  $F$  runs over finite subsets of  $X$ . To prevent any confusion and to distinguish between the vector-valued spectrum and the usual spectrum, we write  $\vec{\text{SP}}(f)$  to denote the vector-valued spectrum.

It is inferred from (1.5) that a function  $\lambda : X \rightarrow \mathbb{C}$  does not belong to the vector-valued spectrum  $\vec{\text{SP}}(f)$  if and only if there exist a finite set  $F = \{x_1, \dots, x_n\}$  in  $X$  and vectors  $a_1, \dots, a_n$  in  $A$  such that

$$\mathbf{1} = \sum_{i=1}^n a_i(\lambda(x_i)\mathbf{1} - f(x_i)). \quad (1.6)$$

Extending (1.4), later in Theorem 3.1, we establish the following equality;

$$\vec{\text{SP}}(f) = \{\phi \circ f : \phi \in \mathfrak{M}(A)\}. \quad (1.7)$$

It is then clear that if  $f : X \rightarrow A$  is continuous, then  $\vec{\text{SP}}(f) \subset C(X)$ . In this case, we will see (Theorem 3.3) that  $\vec{\text{SP}}(f)$  is a compact subset of  $C(X)$ . In general, assume that  $X$  is enriched with some structure (topological, algebraical, etc.), and that  $f : X \rightarrow A$  is an appropriate morphism. The following states that many structural properties of  $f$  are inherited by every  $\lambda \in \vec{\text{SP}}(f)$ .

**Proposition 1.1** ([7], [13]). *Let  $f : X \rightarrow A$  be a function and  $\lambda : X \rightarrow \mathbb{C}$  be in the vector-valued spectrum  $\vec{\text{SP}}(f)$ .*

- (1) *If  $f$  is bounded, then so is  $\lambda$ .*
- (2) *If  $X$  is a topological space and  $f \in C(X, A)$ , then  $\lambda \in C(X)$ .*
- (3) *If  $X \subset \mathbb{C}^n$  and  $f \in H_0(X, A)$ , i.e.  $f$  is holomorphic on a neighbourhood of  $X$ , then  $\lambda \in H_0(X)$ .*
- (4) *If  $(X, d)$  is a metric space and  $f \in \text{Lip}(X, A)$ , then  $\lambda \in \text{Lip}(X)$ .*
- (5) *If  $X = \mathbb{N}$  and  $f \in \ell^1(\mathbb{N}, A)$ , then  $\lambda \in \ell^1(\mathbb{N})$ .*
- (6) *If  $X$  is a linear space and  $f$  is linear, then so is  $\lambda$ .*
- (7) *If  $X$  is a Banach space and  $f \in \mathcal{B}(X, A)$ , then  $\lambda \in X^*$  and  $\|\lambda\| \leq \|f\|$ .*
- (8) *If  $X = \mathfrak{A}$  is a Banach algebra and  $f$  is an algebra homomorphism, then  $\lambda \in \mathfrak{M}(\mathfrak{A})$ .*
- (9) *If  $I : A \rightarrow A$  is the identity operator, then  $\vec{\text{SP}}(I) = \mathfrak{M}(A)$ .*

**1.2. The  $A$ -valued Spectrum.** We take a different approach to studying vector-valued spectrum. To provide a motivation, assume  $X$  is a compact Hausdorff space and  $\mathfrak{A}$  is a complex function algebra on  $X$ . For every  $x \in X$ , the evaluation homomorphism  $\varepsilon_x : \mathfrak{A} \rightarrow \mathbb{C}$ ,  $f \mapsto f(x)$ , is a character of  $\mathfrak{A}$  whence the spectrum  $\text{SP}(f)$  contains the range  $f(X)$ . In case  $\mathfrak{A}$  is a natural function algebra, i.e.  $\varepsilon_x$  ( $x \in X$ ) are the only characters of  $\mathfrak{A}$ , we have the equality  $\text{SP}(f) = f(X)$ .

If  $f : X \rightarrow A$  is an element of a Banach  $A$ -valued function algebra  $\mathscr{A}$ , an  $A$ -valued spectrum  $\vec{\text{SP}}_A(f)$  of  $f$  will be defined in such a way that  $f(X) \subset \vec{\text{SP}}_A(f)$ . In Theorem 4.2, by utilizing  $A$ -valued characters  $\Psi : \mathscr{A} \rightarrow A$  (defined in [1]) the following analogy of (1.1) will be established:

$$\vec{\text{SP}}_A(f) = \{\Psi(f) : \Psi \text{ is an } A\text{-character of } \mathscr{A}\}. \quad (1.8)$$

For example, the only  $A$ -characters of  $\mathscr{A} = C(X, A)$  are the evaluation homomorphisms  $\mathcal{E}_x : C(X, A) \rightarrow A$ ,  $f \mapsto f(x)$ , with  $x \in X$ . Therefore, for all  $f \in \mathscr{A}$ ,

$$\vec{\text{SP}}_A(f) = \{\mathcal{E}_x(f) : x \in X\} = \{f(x) : x \in X\} = f(X).$$

**1.3. Historical Background.** Many different spectra have been defined over the last seventy years. The classical definition was given and developed in the 1950's by Arens and Calderón [3], Silov [18] and Waelbroeck [19] for  $n$ -tuples in a commutative unital Banach algebra. In 1971, Waelbroeck [20] was the first to consider the joint spectrum of an infinite set of elements in a way that led to a functional calculus for norm continuous holomorphic germs, [6]. Subsequently, his results were extended in various ways by Chidami [5] and Matos [14, 15].

When  $A$  is a commutative Banach algebra and  $\mathfrak{A}$  is a Banach space, Waelbroeck defined an  $\mathfrak{A}$ -valued spectrum of an element  $f$  of the projective tensor product  $\mathfrak{A} \widehat{\otimes} A$ . In [14], Matos considered the spectrum as lying in  $C(\mathfrak{M}(A), \mathfrak{A})$ , where  $A$  is a uniform algebra with maximal ideal space  $\mathfrak{M}(A)$  and  $\mathfrak{A}$  is a locally convex space with the approximation property. In [15] he considered the spectrum when  $\mathfrak{A}$  has an unconditional basis and  $A$  is an arbitrary commutative unital Banach algebra. In all these articles an infinite-dimensional holomorphic functional calculus is developed.

In [12], Harte defined, using left and right invertibility, the joint spectrum of a system of elements in an arbitrary Banach algebra and, subject to certain commutativity hypotheses, obtained spectral mapping theorems. In a different setting, Allan [2] obtained a Gelfand invertibility theorem, in an arbitrary unital Banach algebra  $A$ , by imposing internal commutativity conditions on a sufficiently large subalgebra. In [7], Dineen, Harte and Taylor defined a non-commutative version of the Waelbroeck spectrum for tensor product elements in  $\mathfrak{A} \widehat{\otimes}_\gamma A$ , where  $\gamma$  is a uniform cross-norm, and by specialising to the case where  $\mathfrak{A}$  was itself a unital Banach algebra obtained a number of applications. In [8, 9], they continue this investigation and examine the behaviour of the spectrum under polynomials and holomorphic mappings between Banach spaces.

## 2. PRELIMINARIES

**2.1. Notations and Conventions.** In this section, a clear declaration of notations and conventions is given.

Throughout,  $X$  is a compact Hausdorff space and  $A$  is a commutative *semisimple* Banach algebra with identity  $\mathbf{1}$ . The set of invertible elements of  $A$  is denoted by  $\text{Inv}(A)$ . The algebra of all continuous functions  $f : X \rightarrow A$  is denoted by  $C(X, A)$ . The uniform norm of a function  $f \in C(X, A)$  is defined in the obvious way:

$$\|f\|_X = \sup\{\|f(x)\| : x \in X\}.$$

If  $f : X \rightarrow \mathbb{C}$  is a function and  $a \in A$ , we write  $fa$  to denote the  $A$ -valued function  $X \rightarrow A$ ,  $x \mapsto f(x)a$ . If  $\mathfrak{A}$  is an algebra of complex-valued functions on  $X$ , we let  $\mathfrak{A}A$  be the linear span of  $\{fa : f \in \mathfrak{A}, a \in A\}$ . Hence, an element  $f \in \mathfrak{A}A$  is of the form  $f = f_1a_1 + \cdots + f_na_n$  with  $f_j \in \mathfrak{A}$  and  $a_j \in A$ .

Given an element  $a \in A$ , we use the same notation  $a$  for the constant function  $X \rightarrow A$  given by  $a(x) = a$ , for all  $x \in X$ , and consider  $A$  as a closed subalgebra of  $C(X, A)$ . We identify  $\mathbb{C}$  with the closed subalgebra  $\mathbb{C}\mathbf{1}$  of  $A$ . Hence, every  $\mathbb{C}$ -valued function can be seen as an  $A$ -valued function. Given  $f : X \rightarrow \mathbb{C}$ , we use the same notation  $f$  for the function  $X \rightarrow A$ ,  $x \mapsto f(x)\mathbf{1}$ . We regard  $C(X)$  as a closed subalgebra of  $C(X, A)$ .

To every continuous function  $f : X \rightarrow A$ , we correspond the function

$$\tilde{f} : \mathfrak{M}(A) \rightarrow C(X), \quad \tilde{f}(\phi) = \phi \circ f.$$

**2.2. Admissible Vector-valued Function Algebras.** An  $A$ -valued function algebra on  $X$  ([1] or [16]) is a subalgebra  $\mathcal{A}$  of  $C(X, A)$  such that (1)  $\mathcal{A}$  contains the constant functions  $X \rightarrow A$ ,  $x \mapsto a$ , with  $a \in A$ , and (2)  $\mathcal{A}$  separates the points of  $X$ . If  $\mathcal{A}$  is endowed with some submultiplicative norm  $\|\cdot\|$  such that the restriction of  $\|\cdot\|$  to  $A$  is equivalent to the original norm of  $A$ , and  $\|f\|_X \leq \|\tilde{f}\|$ , for every

$f \in \mathcal{A}$ , then  $\mathcal{A}$  is called a *normed  $A$ -valued function algebra* on  $X$ . A complete normed  $A$ -valued function algebra is called a *Banach  $A$ -valued function algebra*. A Banach  $A$ -valued function algebra  $\mathcal{A}$  is called an  *$A$ -valued uniform algebra* if the given norm of  $\mathcal{A}$  is equivalent to the uniform norm  $\|\cdot\|_X$ .

If no confusion can arise, instead of  $\|\cdot\|$ , we use the same notation  $\|\cdot\|$  for the norm of  $\mathcal{A}$ .

**Definition 2.1** ([1]). An  $A$ -valued function algebra  $\mathcal{A}$  is said to be *admissible* if

$$\{\phi \circ f : \phi \in \mathfrak{M}(A), f \in \mathcal{A}\} \subset \mathcal{A}. \quad (2.1)$$

When  $\mathcal{A}$  is admissible, set  $\mathfrak{A} = C(X) \cap \mathcal{A}$ . Then  $\mathfrak{A}$  is the subalgebra of  $\mathcal{A}$  consisting of all complex-valued functions in  $\mathcal{A}$  and forms a complex-valued function algebra by itself.

Admissible vector-valued function algebras are abundant. Some examples are as follows. The algebra  $C(X, A)$  of all continuous  $A$ -valued functions is admissible. If  $\mathfrak{A}$  is a complex function algebra on  $X$  then  $\mathfrak{A}A$  is an admissible  $A$ -valued function algebra on  $X$ , and thus its uniform closure in  $C(X, A)$  is an admissible  $A$ -valued uniform algebra. More generally, the tensor product  $\mathfrak{A} \otimes A$  can be seen as an admissible  $A$ -valued function algebra, and if  $\gamma$  is a cross-norm on  $\mathfrak{A} \otimes A$ , its completion  $\mathfrak{A} \widehat{\otimes}_{\gamma} A$  forms an admissible Banach  $A$ -valued function algebra.

Another example of an admissible Banach  $A$ -valued function algebra is the  $A$ -valued Lipschitz function algebra  $\text{Lip}(X, A)$ , where  $X$  is a compact metric space (Example 5.2). Recently, in [10], the character space and Šilov boundary of  $\text{Lip}(X, A)$  has been studied. An example in [1] shows that not all vector-valued function algebras are admissible.

**2.3. Vector-valued Characters.** Vector-valued characters are an obvious generalization of characters. Let  $\mathcal{A}$  be an  $A$ -valued function algebra on  $X$ . For every  $x \in X$ , define  $\mathcal{E}_x : \mathcal{A} \rightarrow A$  by  $\mathcal{E}_x(f) = f(x)$ . We call  $\mathcal{E}_x$  the *evaluation homomorphism* at the point  $x$ . Point evaluation homomorphisms are good examples of vector-valued characters.

**Definition 2.2** ([1]). Let  $\mathcal{A}$  be an admissible  $A$ -valued function algebra on  $X$ . An  *$A$ -character* of  $\mathcal{A}$  is an algebra homomorphism  $\Psi : \mathcal{A} \rightarrow A$  such that  $\Psi(1) = 1$  and  $\phi(\Psi f) = \Psi(\phi \circ f)$ , for all  $f \in \mathcal{A}$  and  $\phi \in \mathfrak{M}(A)$ . The set of all  $A$ -characters of  $\mathcal{A}$  is denoted by  $\mathfrak{M}_A(\mathcal{A})$ .

If  $\Psi : \mathcal{A} \rightarrow A$  is an  $A$ -character, then  $\psi = \Psi|_{\mathfrak{A}}$  is a character of  $\mathfrak{A}$ . It is proved in [1] that if  $\Psi_1$  and  $\Psi_2$  are  $A$ -characters on  $\mathcal{A}$ , and if  $\psi_1 = \Psi_1|_{\mathfrak{A}}$  and  $\psi_2 = \Psi_2|_{\mathfrak{A}}$ , then

$$\Psi_1 = \Psi_2 \iff \ker \Psi_1 = \ker \Psi_2 \iff \ker \psi_1 = \ker \psi_2 \iff \psi_1 = \psi_2. \quad (2.2)$$

**Definition 2.3** ([1]). Given a character  $\psi \in \mathfrak{M}(\mathfrak{A})$ , if there exists an  $A$ -character  $\Psi$  on  $\mathcal{A}$  such that  $\Psi|_{\mathfrak{A}} = \psi$ , then we say that  $\psi$  *lifts* to the  $A$ -character  $\Psi$ .

By (2.2), if  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to  $\Psi_1$  and  $\Psi_2$ , then  $\Psi_1 = \Psi_2$ . For every  $x \in X$ , the unique  $A$ -character to which the evaluation character  $\varepsilon_x$  lifts is the evaluation homomorphism  $\mathcal{E}_x$ . Conditions under which every character  $\psi$  lifts to some  $A$ -character  $\Psi$  are given in the following.

**Theorem 2.4** ([1]). *Let  $\mathcal{A}$  be an admissible Banach  $A$ -valued function algebra on  $X$ , and let  $E$  be the linear span of  $\mathfrak{M}(A)$  in  $A^*$ . The following statements are equivalent.*

- (i) *For every  $\psi \in \mathfrak{M}(\mathfrak{A})$  and  $f \in \mathcal{A}$ , the mapping  $g : E \rightarrow \mathbb{C}$ , defined by  $g(\phi) = \psi(\phi \circ f)$ , is continuous with respect to the weak\* topology of  $E$ .*

- (ii) Every  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to an  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ .
- (iii) Every  $f \in \mathcal{A}$  has a unique extension  $F : \mathfrak{M}(\mathfrak{A}) \rightarrow A$  such that

$$\phi(F(\psi)) = \psi(\phi \circ f) \quad (\psi \in \mathfrak{M}(\mathfrak{A}), \phi \in \mathfrak{M}(A)).$$

**Theorem 2.5** ([1]). *Let  $\mathcal{A}$  be an admissible Banach  $A$ -valued function algebra on  $X$  and  $\mathfrak{A} = C(X) \cap \mathcal{A}$ . If  $\|\hat{f}\| = \|f\|_X$ , for all  $f \in \mathfrak{A}$ , then every character  $\psi : \mathfrak{A} \rightarrow \mathbb{C}$  lifts to some  $A$ -character  $\Psi : \mathcal{A} \rightarrow A$ . In particular, if  $\mathfrak{A}$  is a uniform algebra, then  $\mathcal{A}$  satisfies all conditions in Theorem 2.4.*

### 3. $C(X)$ -VALUED SPECTRUM

In this section, we study the vector-valued spectrum  $\vec{sP}(f)$  of an  $A$ -valued function  $f$  on  $X$  defined by (1.5)

**Theorem 3.1.**  $\vec{sP}(f) = \{\phi \circ f : \phi \in \mathfrak{M}(A)\}$ .

*Proof.* First, take  $\phi \in \mathfrak{M}(A)$ . We show that  $\phi \circ f \in \vec{sP}(f)$ . If  $\phi \circ f \notin \vec{sP}(f)$ , then, for some points  $x_1, \dots, x_n \in X$  and vectors  $a_1, \dots, a_n \in A$ , we have

$$\mathbf{1} = \sum_{i=1}^n a_i(\phi \circ f(x_i)\mathbf{1} - f(x_i)).$$

Therefore,

$$1 = \phi(\mathbf{1}) = \sum_{i=1}^n \phi(a_i)(\phi(f(x_i)) - \phi(f(x_i))) = 0.$$

This is a contradiction. Hence  $\phi \circ f \in \vec{sP}(f)$ .

Conversely, take  $\lambda \in \vec{sP}(f)$ , and consider the ideal  $I$  in  $A$  generated by

$$S = \{\lambda(x)\mathbf{1} - f(x) : x \in X\}.$$

Since  $\mathbf{1} \notin I$ , the ideal  $I$  is proper. So there is a character  $\phi \in \mathfrak{M}(A)$  with  $I \subset \ker \phi$ . This means that  $\phi(\lambda(x)\mathbf{1} - f(x)) = 0$ , for all  $x \in X$ , whence  $\lambda = \phi \circ f$ .  $\square$

Next, we show that  $\vec{sP}(f)$ , for  $f \in C(X, A)$ , is a compact subset of  $C(X)$ . To this end, the following lemma is needed. Recall that a family  $\mathcal{F}$  of continuous functions from  $X$  into a metric space  $(Y, d)$  is said to be *equicontinuous* if, for every  $x \in X$  and every  $\varepsilon > 0$ , there is a neighborhood  $U_x$  of  $x \in X$  such that

$$d(f(x), f(y)) < \varepsilon \quad (y \in U_x, f \in \mathcal{F}).$$

**Lemma 3.2.** *If a family  $\mathcal{F}$  of functions in  $C(X, A)$  is equicontinuous, then the family  $\mathcal{G}$  defined by*

$$\mathcal{G} = \{\phi \circ f : \phi \in \mathfrak{M}(A), f \in \mathcal{F}\} \tag{3.1}$$

*is equicontinuous.*

*Proof.* Let  $x_0 \in X$  and  $\varepsilon > 0$ . Since  $\mathcal{F}$  is equicontinuous, there is a neighbourhood  $U_0$  of  $x_0$  such that  $\|f(x) - f(x_0)\| < \varepsilon$ , for all  $f \in \mathcal{F}$  and  $x \in U_0$ . Then

$$|\phi \circ f(x) - \phi \circ f(x_0)| \leq \|f(x) - f(x_0)\| < \varepsilon \quad (\phi \in \mathfrak{M}(A), f \in \mathcal{F}, x \in U_0).$$

Hence  $\mathcal{G}$  is equicontinuous.  $\square$

The Arzelà-Ascoli theorem states that a family  $\mathcal{F} \subset C(X)$  is relatively compact, in the topology induced by the uniform norm, if it is equicontinuous and pointwise bounded.

**Theorem 3.3.** *Suppose  $f : X \rightarrow A$  is continuous. Then  $\vec{sP}(f)$  is a compact subset of  $C(X)$ .*

*Proof.* That  $\vec{sP}(f) \subset C(X)$  follows from Theorem 3.1 and the assumption that  $f : X \rightarrow A$  is continuous. To show that  $\vec{sP}(f)$  is compact in  $C(X)$ , using Arzelà-Ascoli theorem, we only need to show that  $\vec{sP}(f)$  is uniformly closed, uniformly bounded and equicontinuous.

We prove that  $\vec{sP}(f)$  is uniformly closed in  $C(X)$ . Take a function  $\lambda \in C(X)$  and assume that  $\lambda \notin \vec{sP}(f)$ . Then, there exist  $a_1, \dots, a_m \in A$  and  $x_1, \dots, x_m \in X$  such that

$$\mathbf{1} = \sum_{i=1}^m a_i (\lambda(x_i) \mathbf{1} - f(x_i)). \quad (3.2)$$

To get a contradiction, assume  $\lambda \in \overline{\vec{sP}(f)}$ . Then, there is a sequence  $\{\phi_n\}$  in  $\mathfrak{M}(A)$  such that  $\phi_n \circ f \rightarrow \lambda$ , uniformly on  $X$ . Take  $\varepsilon = (\|a_1\| + \dots + \|a_m\|)^{-1}$ , where  $a_1, \dots, a_m$  are given by (3.2). There is  $N \in \mathbb{N}$ , such that  $\|\phi_n \circ f - \lambda\|_X < \varepsilon$ , for all  $n \geq N$ . This implies that

$$|\phi_N(f(x_i)) - \lambda(x_i)| < \varepsilon \quad (i = 1, 2, \dots, m).$$

Now, by (3.2), we get

$$\mathbf{1} = \phi_N(\mathbf{1}) = \sum_{i=1}^m \phi_N(a_i) (\lambda(x_i) - \phi_N(f(x_i))).$$

Therefore,

$$1 \leq \sum_{i=1}^m |\phi_N(a_i) (\lambda(x_i) - \phi_N(f(x_i)))| < \sum_{i=1}^m \|a_i\| \varepsilon = 1.$$

This is a contradiction, and thus  $\lambda \notin \overline{\vec{sP}(f)}$ . We conclude that  $\vec{sP}(f)$  is uniformly closed in  $C(X)$ . That  $\vec{sP}(f)$  is uniformly bounded follows from the fact that, for every  $\phi \in \mathfrak{M}(A)$ ,

$$\|\phi \circ f\|_X = \sup\{|\phi(f(x))| : x \in X\} \leq \|\phi\| \sup\{\|f(x)\| : x \in X\} = \|f\|_X.$$

Finally, by taking  $\mathcal{F} = \{f\}$ , Lemma 3.2 implies that  $\vec{sP}(f)$  is equicontinuous.

The set  $\vec{sP}(f)$  being uniformly closed, uniformly bounded and equicontinuous is a compact subset of  $C(X)$ .  $\square$

Now, analogous to [4, Proposition 5.17], we prove that the set-valued mapping  $f \mapsto \vec{sP}(f)$  of  $\mathcal{A}$  into the family of compact sets in  $C(X)$  is upper semi-continuous.

**Lemma 3.4.** *Suppose  $f_n \rightarrow f$  in  $\mathcal{A}$ ,  $\lambda_n \in \vec{sP}(f_n)$  and  $\lambda_n \rightarrow \lambda$  in  $C(X)$ . Then  $\lambda \in \vec{sP}(f)$ .*

*Proof.* Towards a contradiction, assume  $\lambda \notin \vec{sP}(f)$ . Then there exists a finite set of points  $x_1, \dots, x_n$  in  $X$  and vectors  $a_1, \dots, a_n$  in  $A$  such that

$$\mathbf{1} = \sum_{i=1}^n a_i (\lambda(x_i) \mathbf{1} - f(x_i)). \quad (3.3)$$

Take  $\varepsilon = 1/(2 \sum \|a_i\|)$ . By the assumption, there is  $N \in \mathbb{N}$  such that

$$\|f_N - f\|_X < \varepsilon, \quad \|\lambda_N - \lambda\|_X < \varepsilon.$$

Since  $\lambda_N \in \vec{\text{SP}}(f_N)$ , by Theorem 3.1, there is some  $\phi \in \mathfrak{M}(A)$  such that  $\lambda_N = \phi \circ f_N$ . Using (3.3), we have

$$\begin{aligned} 1 &= \left| \sum_{i=1}^n \phi(a_i) (\lambda(x_i) - \phi \circ f(x_i)) \right| \\ &\leq \sum_{i=1}^n \|a_i\| |\lambda(x_i) - \lambda_N(x_i) + \phi \circ f_N(x_i) - \phi \circ f(x_i)| \\ &\leq \sum_{i=1}^n \|a_i\| (\|\lambda - \lambda_N\|_X + \|f_N - f\|_X) \\ &< 2\varepsilon \sum_{i=1}^n \|a_i\| = 1. \end{aligned}$$

This is absurd. Hence  $\lambda \in \vec{\text{SP}}(f)$ .  $\square$

**Theorem 3.5.** *The mapping  $f \mapsto \vec{\text{SP}}(f)$  of  $\mathcal{A}$  into the family of compact sets in  $C(X)$  is upper semi-continuous.*

*Proof.* Suppose the mapping is not upper semi-continuous at some  $f_0 \in \mathcal{A}$ . Hence, there is a neighbourhood  $\Omega$  of  $\vec{\text{SP}}(f_0)$  in  $C(X)$  and a sequence  $\{f_n\}$  in  $\mathcal{A}$  such that  $f_n \rightarrow f_0$  and  $\vec{\text{SP}}(f_n) \not\subset \Omega$ . Set  $\mathcal{F} = \{f_0, f_1, f_2, \dots\}$  and define  $\mathcal{G}$  as in (3.1). Since  $f_n \rightarrow f_0$  uniformly on  $X$ , the family  $\mathcal{F}$  is equicontinuous on  $X$ . By Lemma 3.2, the family  $\mathcal{G}$  is equicontinuous. Since  $\mathcal{F}$  is uniformly bounded,  $\mathcal{G}$  is uniformly bounded, and, therefore, relatively compact.

Now, consider the mapping  $\vec{\text{SP}} : \mathcal{F} \rightarrow 2^{\mathcal{G}}$ ,  $f \mapsto \vec{\text{SP}}(f)$ . By Lemma 3.4 and [4, Lemma 5.16], this mapping is upper semi-continuous at  $f_0$ , which contradicts our primary assumption.  $\square$

**The  $\mathfrak{A}$ -valued spectrum.** When  $\mathcal{A}$  is an admissible  $A$ -valued function algebra on  $X$  and  $\mathfrak{A} = \mathcal{A} \cap C(X)$ , then  $\vec{\text{SP}}(f) \subset \mathfrak{A}$ , for every  $f \in \mathcal{A}$ . In this case, it is convenient to denote the vector-valued spectrum by  $\vec{\text{SP}}_{\mathfrak{A}}(f)$  and call it the  $\mathfrak{A}$ -valued spectrum. If the mapping  $\tilde{f} : \mathfrak{M}(A) \rightarrow \mathfrak{A}$ ,  $\phi \mapsto \phi \circ f$ , is continuous (which is the case when  $\mathfrak{A}$  is a uniform algebra) then  $\vec{\text{SP}}_{\mathfrak{A}}(f)$  is compact in  $\mathfrak{A}$  since  $\vec{\text{SP}}_{\mathfrak{A}}(f) = \tilde{f}(\vec{\text{SP}}(A))$ . In general, however, it is unknown if  $\tilde{f}$  is continuous or  $\vec{\text{SP}}_{\mathfrak{A}}(f)$  is compact in  $\mathfrak{A}$ . We remark that  $\vec{\text{SP}}_{\mathfrak{A}}(f)$  is always closed in  $\mathfrak{A}$ .

#### 4. THE $A$ -VALUED SPECTRUM

Let  $\mathcal{A}$  be an admissible Banach  $A$ -valued function algebra on  $X$ . Take  $f \in \mathcal{A}$  and consider the function  $\tilde{f} : \mathfrak{M}(A) \rightarrow \mathfrak{A}$ ,  $\phi \mapsto \phi \circ f$ . By Theorem 3.1, we have

$$\vec{\text{SP}}(\tilde{f}) = \{\psi \circ \tilde{f} : \psi \in \mathfrak{M}(\mathfrak{A})\}.$$

Note that every  $\lambda \in \vec{\text{SP}}(\tilde{f})$  is a function of  $\mathfrak{M}(A)$  into  $\mathbb{C}$ . Also, every  $a \in A$  induces a function  $\hat{a} : \mathfrak{M}(A) \rightarrow \mathbb{C}$ , the Gelfand transform of  $a$ .

**Definition 4.1.** The  $A$ -valued spectrum of a function  $f \in \mathcal{A}$  is defined by

$$\vec{\text{SP}}_A(f) = \{a \in A : \hat{a} \in \vec{\text{SP}}(\tilde{f})\}. \quad (4.1)$$

**Theorem 4.2.** *For every  $f \in \mathcal{A}$ , we have*

$$f(X) \subset \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathcal{A})\} \subset \vec{\text{SP}}_A(f). \quad (4.2)$$

*If every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ , then*

$$\vec{\text{SP}}_A(f) = \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathcal{A})\}. \quad (4.3)$$

Equality (4.3) is an analogy of (1.1).

*Proof.* The first inclusion in (4.2) is obvious since, for every  $x \in X$ , the evaluation homomorphism  $\mathcal{E}_x : f \mapsto f(x)$  is an  $A$ -character and  $f(X) = \{\mathcal{E}_x(f) : x \in X\}$ .

To prove the second inclusion in (4.2), take an  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$  and let  $a = \Psi(f)$ . Take a finite set  $\phi_1, \dots, \phi_n$  of elements in  $\mathfrak{M}(A)$  and  $g_1, \dots, g_n$  of functions in  $\mathfrak{A}$ , and consider the function

$$g = \sum_{i=1}^n g_i(\hat{a}(\phi_i)\mathbf{1} - \phi_i \circ f).$$

Let  $\psi = \Psi|_{\mathfrak{A}}$ . Then  $\psi \in \mathfrak{M}(\mathfrak{A})$  and thus  $\psi(\mathbf{1}) = 1$ . We have

$$\begin{aligned} \psi(g) &= \psi\left(\sum_{i=1}^n g_i(\hat{a}(\phi_i)\mathbf{1} - \phi_i \circ f)\right) = \sum_{i=1}^n \psi(g_i)(\hat{a}(\phi_i) - \psi(\phi_i \circ f)) \\ &= \sum_{i=1}^n \psi(g_i)(\phi_i(a) - \phi_i(\Psi f)) \\ &= \sum_{i=1}^n \psi(g_i)(\phi_i(a) - \phi_i(a)) = 0. \end{aligned}$$

This implies that  $g \neq \mathbf{1}$ . Hence  $a \in \vec{sP}_A(f)$ , that is,  $\Psi(f) \in \vec{sP}_A(f)$ .

Now, assume that every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ . Take an element  $a \in \vec{sP}_A(f)$ . We show the existence of some  $A$ -character  $\Psi$  such that  $a = \Psi(f)$ . Since  $a \in \vec{sP}_A(f)$ , we have  $\hat{a} \in \vec{sP}(\tilde{f})$ . Hence there is a character  $\psi \in \mathfrak{M}(\mathfrak{A})$  such that  $\hat{a} = \psi \circ \tilde{f}$ . Assume  $\psi$  lifts to the  $A$ -character  $\Psi$  of  $\mathcal{A}$ . Then, for every  $\phi \in \mathfrak{M}(A)$ ,

$$\phi(a) = \hat{a}(\phi) = \psi(\tilde{f}(\phi)) = \psi(\phi \circ f) = \phi(\Psi(f)).$$

Since  $A$  is assumed to be semisimple, we get  $a = \Psi(f)$ . □

*Remark.* If every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ , then by Theorem 2.4 each  $f \in \mathcal{A}$  extends to a function  $F : \mathfrak{M}(\mathfrak{A}) \rightarrow A$ . In this case,  $\vec{sP}_A(f) = F(\mathfrak{M}(\mathfrak{A}))$  which is compact if  $F$  is continuous.

**Corollary 4.3.** *If  $\mathfrak{A} = C(X) \cap \mathcal{A}$  is a natural function algebra on  $X$ , then  $\vec{sP}_A(f) = f(X)$ , for all  $f \in \mathcal{A}$ . In particular,  $\vec{sP}_A(f)$  is compact in  $A$ .*

*Proof.* Since  $\mathfrak{A}$  is natural, the only characters of  $\mathfrak{A}$  are the point evaluation homomorphisms  $\varepsilon_x$  ( $x \in X$ ). Therefore,  $\mathcal{E}_x$  ( $x \in X$ ) are the only  $A$ -characters of  $\mathcal{A}$ . From (4.3), we get  $\vec{sP}_A(f) = f(X)$  for all  $f \in \mathcal{A}$ . □

## 5. EXAMPLES

In this section, we identify the  $A$ -valued spectrum in some well-known algebras.

**Example 5.1.** Let  $\mathcal{A} = C(X, A)$ . Then  $\mathfrak{A} = C(X)$  is natural. By Corollary 4.3, we have  $\vec{sP}_A(f) = f(X)$ , for all  $f \in C(X, A)$ .

**Example 5.2.** Let  $(X, \rho)$  be a compact metric space. An  $A$ -valued *Lipschitz function* is a function  $f : X \rightarrow A$  such that

$$L(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{\rho(x, y)} : x, y \in X, x \neq y \right\} < \infty. \quad (5.1)$$

Denoted by  $\text{Lip}(X, A)$ , the space of  $A$ -valued Lipschitz functions on  $X$  is an  $A$ -valued function algebra on  $X$ , called the  *$A$ -valued Lipschitz algebra*. For  $f \in \text{Lip}(X, A)$ , the Lipschitz norm of  $f$  is defined by  $\|f\|_L = \|f\|_X + L(f)$ . It is easily verified that  $(\text{Lip}(X, A), \|\cdot\|_L)$  is an admissible Banach  $A$ -valued function algebra



and  $\text{Lip}(X) = \text{Lip}(X, A) \cap C(X)$ , where  $\text{Lip}(X) = \text{Lip}(X, \mathbb{C})$  is the classic complex-valued Lipschitz algebra. It is proved in [17] (see also [10]) that  $\text{Lip}(X)$  is natural. Therefore, by Corollary 4.3, we have  $\bar{s}\vec{P}_A(f) = f(X)$ , for all  $f \in \text{Lip}(X, A)$ .

**Example 5.3.** Let  $K$  be a compact set in the complex plane. Let  $P_0(K, A)$  be the algebra of the restriction to  $K$  of all polynomials  $p(z) = a_n z^n + \cdots + a_1 z + a_0$  with coefficients  $a_0, a_1, \dots, a_n$  in  $A$ . Let  $R_0(K, A)$  be the algebra of the restriction to  $K$  of all rational functions of the form  $p(z)/q(z)$ , where  $p(z)$  and  $q(z)$  are polynomials with coefficients in  $A$ , and  $q(\lambda) \in \text{Inv}(A)$ , whenever  $\lambda \in K$ .

The algebras  $P_0(K, A)$  and  $R_0(K, A)$  are  $A$ -valued function algebras on  $K$ , and their uniform closures in  $C(K, A)$ , denoted by  $P(K, A)$  and  $R(K, A)$ , are  $A$ -valued uniform algebras. When  $A = \mathbb{C}$ , we drop  $A$  and write  $P(K)$  and  $R(K)$ , which are complex uniform algebras.

For  $P(K, A)$ , we have  $\mathfrak{A} = P(K)$  and it is well-known that the character space of  $P(K)$  is naturally identified with  $\hat{K}$  the polynomially convex hull of  $K$  ([11]). Since  $\mathfrak{A}$  is a uniform algebra, by Theorems 2.4 and 2.5, every  $f \in P(K, A)$  extends to a function  $F : \hat{K} \rightarrow A$ . We see that, for all  $f \in P(K, A)$ ,

$$\bar{s}\vec{P}_A(f) = \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathcal{A})\} = \{F(\psi) : \psi \in \mathfrak{M}(\mathfrak{A})\} = F(\hat{K})$$

For  $R(K, A)$ , we have  $\mathfrak{A} = R(K)$  and it is well-known that  $R(K)$  is natural ([11]). By Corollary 4.3, we have

$$\bar{s}\vec{P}_A(f) = f(K) \quad (f \in R(K, A)).$$

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# VECTOR-VALUED SPECTRA OF BANACH ALGEBRA VALUED CONTINUOUS FUNCTIONS

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**ABSTRACT.** Given a compact space  $X$ , a commutative Banach algebra  $A$ , and an  $A$ -valued function algebra  $\mathcal{A}$  on  $X$ , the notions of vector-valued spectrum of functions  $f \in \mathcal{A}$  are discussed. The  $A$ -valued spectrum  $\vec{\text{SP}}_A(f)$  of every  $f \in \mathcal{A}$  is defined in such a way that  $f(X) \subset \vec{\text{SP}}_A(f)$ . Utilizing the  $A$ -characters introduced in (M. Abtahi, *Vector-valued characters on vector-valued function algebras*, Banach J. Math. Anal. (to appear) [arXiv:1509.09215](#)), it is proved that  $\vec{\text{SP}}_A(f) = \{\Psi(f) : \Psi \text{ is an } A\text{-character of } \mathcal{A}\}$ . For the so-called natural  $A$ -valued function algebras, such as  $C(X, A)$  and  $\text{Lip}(X, A)$ , we see that  $\vec{\text{SP}}_A(f) = f(X)$ . When  $A = \mathbb{C}$ , Banach  $A$ -valued function algebras reduce to Banach function algebras,  $A$ -characters reduce to characters, and  $A$ -valued spectra reduce to usual spectra.

## 1. INTRODUCTION

Let  $A$  be a commutative complex Banach algebra with a unit element  $\mathbf{1}$ , and let  $\mathfrak{M}(A)$  denote the character space (maximal ideal space) of  $A$ . Given  $a \in A$ , the spectrum  $\text{SP}(a)$  of  $a$  consists of those complex numbers  $\lambda$  for which  $\lambda\mathbf{1} - a$  is not invertible in  $A$ . It is known that  $\text{SP}(a)$  is compact and

$$\text{SP}(a) = \{\phi(a) : \phi \in \mathfrak{M}(A)\}. \quad (1.1)$$

**1.1. Vector-valued Spectra.** Let  $n$  be a positive integer and  $\mathbf{a} = (a_1, \dots, a_n)$  be an  $n$ -tuples of elements of  $A$ . The *joint spectrum* of  $\mathbf{a}$ , again denoted by  $\text{SP}(\mathbf{a})$ , is defined to be the set of all  $n$ -tuples  $(\lambda_1, \dots, \lambda_n)$  of scalars in  $\mathbb{C}$  such that the unit element  $\mathbf{1}$  does not belong to the ideal generated of  $A$  by  $\{\lambda_i\mathbf{1} - a_i : 1 \leq i \leq n\}$ . That is,

$$\text{SP}(\mathbf{a}) = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \mathbf{1} \notin \sum_{i=1}^n A(\lambda_i\mathbf{1} - a_i) \right\}. \quad (1.2)$$

It is proved (e.g. [8]) that  $\text{SP}(\mathbf{a})$  is a compact set in  $\mathbb{C}^n$  and

$$\text{SP}(\mathbf{a}) = \{(\phi(a_1), \dots, \phi(a_n)) : \phi \in \mathfrak{M}(A)\}. \quad (1.3)$$

If one sets  $X = \{1, \dots, n\}$ , every  $n$ -tuples  $\mathbf{a} = (a_1, \dots, a_n)$  can be seen as an  $A$ -valued function  $\mathbf{a} : X \rightarrow A$ ,  $i \mapsto a_i$ . For every  $\phi \in \mathfrak{M}(A)$ , the composition of

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$\mathbf{a} : X \rightarrow A$  and  $\phi : A \rightarrow \mathbb{C}$  gives  $\phi \circ \mathbf{a} : X \rightarrow \mathbb{C}$ ,  $i \mapsto \phi(a_i)$ . With this convention, equality (1.3) is rephrased as follows;

$$\text{SP}(\mathbf{a}) = \{\phi \circ \mathbf{a} : \phi \in \mathfrak{M}(A)\}. \quad (1.4)$$

In general, let  $X$  be a nonempty set, and let  $f : X \rightarrow A$  be a function. The *vector-valued spectrum* of  $f$  is defined to be

$$\vec{\text{SP}}(f) = \left\{ \lambda : X \rightarrow \mathbb{C} : \mathbf{1} \notin \sum_{x \in F} A(\lambda(x)\mathbf{1} - f(x)) \right\}, \quad (1.5)$$

where  $F$  runs over finite subsets of  $X$ . To prevent any confusion and to distinguish between different types of spectrum, vector-valued spectrum is denoted  $\vec{\text{SP}}(f)$ .

Form (1.5) a function  $\lambda : X \rightarrow \mathbb{C}$  does not belong to  $\vec{\text{SP}}(f)$  if, and only if, there exist  $x_1, \dots, x_n$  in  $X$  and  $a_1, \dots, a_n$  in  $A$  such that

$$\mathbf{1} = \sum_{i=1}^n a_i(\lambda(x_i)\mathbf{1} - f(x_i)). \quad (1.6)$$

Extending (1.4), later in Theorem 3.1, we establish the following equality;

$$\vec{\text{SP}}(f) = \{\phi \circ f : \phi \in \mathfrak{M}(A)\}. \quad (1.7)$$

It is then clear that if  $f : X \rightarrow A$  is continuous, then  $\vec{\text{SP}}(f) \subset C(X)$ . In this case, we will see (Theorem 3.3) that  $\vec{\text{SP}}(f)$  is a compact subset of  $C(X)$ . In general, if  $X$  is enriched with some structure (topological, algebraical, etc.), and  $f : X \rightarrow A$  is an appropriate morphism, then many structural properties of  $f$  are inherited by every  $\lambda \in \vec{\text{SP}}(f)$ .

**Proposition 1.1** ([5], [10]). *Let  $f : X \rightarrow A$  be a function and  $\lambda : X \rightarrow \mathbb{C}$  be in the vector-valued spectrum  $\vec{\text{SP}}(f)$ .*

- (1) *If  $f$  is bounded, then so is  $\lambda$ .*
- (2) *If  $X$  is a topological space and  $f \in C(X, A)$ , then  $\lambda \in C(X)$ .*
- (3) *If  $X \subset \mathbb{C}^n$  and  $f \in H_0(X, A)$ , i.e.  $f$  is holomorphic on a neighbourhood of  $X$ , then  $\lambda \in H_0(X)$ .*
- (4) *If  $(X, d)$  is a metric space and  $f \in \text{Lip}(X, A)$ , then  $\lambda \in \text{Lip}(X)$ .*
- (5) *If  $X = \mathbb{N}$  and  $f \in \ell^1(\mathbb{N}, A)$ , then  $\lambda \in \ell^1(\mathbb{N})$ .*
- (6) *If  $X$  is a linear space and  $f$  is linear, then so is  $\lambda$ .*
- (7) *If  $X$  is a Banach space and  $f \in \mathcal{B}(X, A)$ , then  $\lambda \in X^*$  and  $\|\lambda\| \leq \|f\|$ .*
- (8) *If  $X = \mathfrak{A}$  is a Banach algebra and  $f$  is an algebra homomorphism, then  $\lambda \in \mathfrak{M}(\mathfrak{A})$ .*
- (9) *If  $I : A \rightarrow A$  is the identity operator, then  $\vec{\text{SP}}(I) = \mathfrak{M}(A)$ .*

**1.2. The  $A$ -valued Spectrum.** We take a different approach to studying vector-valued spectrum. To provide a motivation, assume  $\mathfrak{A}$  is a complex function algebra on  $X$ . For every  $x \in X$ , the evaluation homomorphism  $\varepsilon_x : f \mapsto f(x)$  is a character of  $\mathfrak{A}$  whence the spectrum  $\text{SP}(f)$  contains the range  $f(X)$ . In case  $\mathfrak{A}$  is natural, i.e.  $\varepsilon_x$  ( $x \in X$ ) are the only characters of  $\mathfrak{A}$ , we get  $\text{SP}(f) = f(X)$ .

Let  $\mathcal{A}$  be an  $A$ -valued function algebra on  $X$ . For every  $f \in \mathcal{A}$ , an  $A$ -valued spectrum  $\vec{\text{SP}}_A(f)$  of  $f$  will be defined in such a way that  $f(X) \subset \vec{\text{SP}}_A(f)$ . Utilizing the  $A$ -valued characters [1], the following analogy of (1.1) will be established:

$$\vec{\text{SP}}_A(f) = \{\Psi(f) : \Psi \text{ is an } A\text{-character of } \mathcal{A}\}. \quad (1.8)$$

**1.3. Historical Background.** Many different spectra have been defined over the last seventy years. The classical definition was given and developed in the 1950's by Arens and Calderón [2], Silov [15] and Waelbroeck [16] for  $n$ -tuples in a commutative unital Banach algebra. Waelbroeck was the first to consider the joint spectrum of an infinite set of elements in a way that led to a functional calculus for norm continuous holomorphic germs, [4]. Subsequently, his results were extended in various ways by Matos [12, 13] and others.

When  $A$  is a commutative Banach algebra and  $\mathfrak{A}$  is a Banach space, Waelbroeck defined an  $\mathfrak{A}$ -valued spectrum of an element  $f$  of the projective tensor product  $\mathfrak{A} \hat{\otimes} A$ . In [12], Matos considered the spectrum as lying in  $C(\mathfrak{M}(A), \mathfrak{A})$ , where  $A$  is a uniform algebra with maximal ideal space  $\mathfrak{M}(A)$  and  $\mathfrak{A}$  is a locally convex space with the approximation property. In [13] he considered the spectrum when  $\mathfrak{A}$  has an unconditional basis and  $A$  is an arbitrary commutative unital Banach algebra. In all these articles an infinite-dimensional holomorphic functional calculus is developed.

In [9], Harte defined, using left and right invertibility, the joint spectrum of a system of elements in an arbitrary Banach algebra and, subject to certain commutativity hypotheses, obtained spectral mapping theorems. In [5], Dineen, Harte and Taylor defined a non-commutative version of the Waelbroeck spectrum for tensor product elements in  $\mathfrak{A} \hat{\otimes}_\gamma A$ , where  $\gamma$  is a uniform cross-norm, and by specialising to the case where  $\mathfrak{A}$  was itself a unital Banach algebra obtained a number of applications. In [6, 7], they continue this investigation and examine the behaviour of the spectrum under polynomials and holomorphic mappings between Banach spaces.

## 2. PRELIMINARIES

**2.1. Notations and Conventions.** Throughout,  $X$  is a compact Hausdorff space and  $A$  is a commutative *semisimple* Banach algebra with a unit element **1**. The set of invertible elements of  $A$  is denoted by  $\text{Inv}(A)$ . The algebra of all continuous  $A$ -valued functions is denoted by  $C(X, A)$ . The uniform norm  $\|f\|_X$  of a function  $f \in C(X, A)$  is defined in the obvious way.

If  $f : X \rightarrow \mathbb{C}$  is a function and  $a \in A$ , we write  $fa$  to denote the  $A$ -valued function  $X \rightarrow A$ ,  $x \mapsto f(x)a$ . If  $\mathfrak{A}$  is an algebra of complex-valued functions on  $X$ , we let  $\mathfrak{A}A$  be the linear span of  $\{fa : f \in \mathfrak{A}, a \in A\}$ . Hence, an element  $f \in \mathfrak{A}A$  is of the form  $f = f_1a_1 + \cdots + f_na_n$  with  $f_j \in \mathfrak{A}$  and  $a_j \in A$ .

Given an element  $a \in A$ , we use the same notation  $a$  for the constant function  $X \rightarrow A$  given by  $a(x) = a$ , for all  $x \in X$ , and consider  $A$  as a closed subalgebra

of  $C(X, A)$ . We identify  $\mathbb{C}$  with the closed subalgebra  $\mathbb{C}\mathbf{1}$  of  $A$ . Hence, every  $\mathbb{C}$ -valued function can be seen as an  $A$ -valued function. Given  $f : X \rightarrow \mathbb{C}$ , we use the same notation  $f$  for the function  $X \rightarrow A$ ,  $x \mapsto f(x)\mathbf{1}$ . We regard  $C(X)$  as a closed subalgebra of  $C(X, A)$ .

To every continuous function  $f : X \rightarrow A$ , we correspond the function

$$\tilde{f} : \mathfrak{M}(A) \rightarrow C(X), \quad \tilde{f}(\phi) = \phi \circ f.$$

**2.2. Admissible Vector-valued Function Algebras.** An  $A$ -valued function algebra on  $X$  is a subalgebra  $\mathcal{A}$  of  $C(X, A)$  that contains the constant functions  $X \rightarrow A$ ,  $x \mapsto a$ , with  $a \in A$ , and separates the points of  $X$ . If  $\mathcal{A}$  is endowed with some complete algebra norm  $\|\cdot\|$  such that the restriction of  $\|\cdot\|$  to  $A$  is equivalent to the original norm of  $A$ , and  $\|f\|_X \leq \|f\|$ , for every  $f \in \mathcal{A}$ , then  $\mathcal{A}$  is called a *Banach  $A$ -valued function algebra* on  $X$ . If the given norm is equivalent to the uniform norm  $\|\cdot\|_X$ ,  $\mathcal{A}$  is called an  *$A$ -valued uniform algebra*. If no confusion can arise, instead of  $\|\cdot\|$ , we use the same notation  $\|\cdot\|$  for the norm of  $\mathcal{A}$ .

**Definition 2.1** ([1]). An  $A$ -valued function algebra  $\mathcal{A}$  is said to be *admissible* if

$$\{(\phi \circ f)\mathbf{1} : \phi \in \mathfrak{M}(A), f \in \mathcal{A}\} \subset \mathcal{A}. \quad (2.1)$$

Admissible vector-valued function algebras are abundant. Some examples are as follows. The algebra  $C(X, A)$  of all continuous  $A$ -valued functions is admissible. If  $\mathfrak{A}$  is a complex function algebra on  $X$  then  $\mathfrak{A}A$  is an admissible  $A$ -valued function algebra on  $X$ , and thus its uniform closure in  $C(X, A)$  is an admissible  $A$ -valued uniform algebra. More generally, the tensor product  $\mathfrak{A} \otimes A$  can be seen as an admissible  $A$ -valued function algebra, and if  $\gamma$  is a cross-norm on  $\mathfrak{A} \otimes A$ , its completion  $\mathfrak{A} \widehat{\otimes}_{\gamma} A$  forms an admissible Banach  $A$ -valued function algebra.

Another example of an admissible Banach  $A$ -valued function algebra is the  $A$ -valued Lipschitz function algebra  $\text{Lip}(X, A)$ , where  $X$  is a compact metric space (Example 5.2). An example in [1] shows that not all vector-valued function algebras are admissible.

During the paper, we let  $\mathcal{A}$  be admissible and  $\mathfrak{A} = \mathcal{A} \cap C(X)$ , more precisely  $\mathfrak{A} = \mathcal{A} \cap C(X)\mathbf{1}$ , be the subalgebra of  $\mathcal{A}$  consisting of all complex-valued functions in  $\mathcal{A}$ . Then  $\mathfrak{A}$  forms a complex-valued function algebra by itself and  $\phi[\mathcal{A}] = \mathfrak{A}$ , for all  $\phi \in \mathfrak{M}(A)$ .

**2.3. Vector-valued Characters.** Vector-valued characters are an obvious generalization of characters. For every  $x \in X$ , define  $\mathcal{E}_x : \mathcal{A} \rightarrow A$  by  $\mathcal{E}_x(f) = f(x)$ . We call  $\mathcal{E}_x$  the *evaluation homomorphism* at the point  $x$ . Point evaluation homomorphisms are good examples of vector-valued characters.

**Definition 2.2** ([1]). An  $A$ -character of  $\mathcal{A}$  is an algebra homomorphism  $\Psi : \mathcal{A} \rightarrow A$  such that  $\Psi(\mathbf{1}) = \mathbf{1}$  and  $\phi(\Psi f) = \Psi(\phi \circ f)$ , for all  $f \in \mathcal{A}$  and  $\phi \in \mathfrak{M}(A)$ . The set of all  $A$ -characters of  $\mathcal{A}$  is denoted by  $\mathfrak{M}_A(\mathcal{A})$ .

If  $\Psi : \mathcal{A} \rightarrow A$  is an  $A$ -character, then  $\psi = \Psi|_{\mathfrak{A}}$  is a character of  $\mathfrak{A}$ . If  $\Psi_1$  and  $\Psi_2$  are  $A$ -characters of  $\mathcal{A}$  with  $\psi_1 = \Psi_1|_{\mathfrak{A}}$  and  $\psi_2 = \Psi_2|_{\mathfrak{A}}$ , then [1]

$$\Psi_1 = \Psi_2 \iff \ker \Psi_1 = \ker \Psi_2 \iff \ker \psi_1 = \ker \psi_2 \iff \psi_1 = \psi_2. \quad (2.2)$$

**Definition 2.3** ([1]). Given a character  $\psi \in \mathfrak{M}(\mathfrak{A})$ , if there exists an  $A$ -character  $\Psi$  on  $\mathcal{A}$  such that  $\Psi|_{\mathfrak{A}} = \psi$ , then we say that  $\psi$  *lifts* to the  $A$ -character  $\Psi$ .

By (2.2), if  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to  $\Psi_1$  and  $\Psi_2$ , then  $\Psi_1 = \Psi_2$ . For every  $x \in X$ , the unique  $A$ -character to which the evaluation character  $\varepsilon_x$  lifts is the evaluation homomorphism  $\mathcal{E}_x$ . Conditions under which every character  $\psi$  lifts to some  $A$ -character  $\Psi$  are given in the following.

**Theorem 2.4** ([1]). *Let  $E$  be the linear span of  $\mathfrak{M}(A)$  in  $A^*$ . The following statements are equivalent.*

- (i) *For every  $\psi \in \mathfrak{M}(\mathfrak{A})$  and  $f \in \mathcal{A}$ , the mapping  $g : E \rightarrow \mathbb{C}$ , defined by  $g(\phi) = \psi(\phi \circ f)$ , is continuous with respect to the weak\* topology of  $E$ .*
- (ii) *Every  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to an  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ .*
- (iii) *Every  $f \in \mathcal{A}$  has a unique extension  $F : \mathfrak{M}(\mathfrak{A}) \rightarrow A$  such that*

$$\phi(F(\psi)) = \psi(\phi \circ f) \quad (\psi \in \mathfrak{M}(\mathfrak{A}), \phi \in \mathfrak{M}(A)).$$

We recall that  $\mathfrak{A} = \mathcal{A} \cap C(X)\mathbf{1}$ .

**Theorem 2.5** ([1]). *If  $\|\hat{f}\| = \|f\|_X$ , for all  $f \in \mathfrak{A}$ , then every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ . In particular, if  $\mathfrak{A}$  is a uniform algebra, then  $\mathcal{A}$  satisfies all conditions in Theorem 2.4.*

### 3. $C(X)$ -VALUED SPECTRUM

In this section, we study the vector-valued spectrum  $\vec{\text{SP}}(f)$  of an  $A$ -valued function  $f$  on  $X$  defined by (1.5)

**Theorem 3.1.**  $\vec{\text{SP}}(f) = \{\phi \circ f : \phi \in \mathfrak{M}(A)\}$ .

*Proof.* First, take  $\phi \in \mathfrak{M}(A)$ . We show that  $\phi \circ f \in \vec{\text{SP}}(f)$ . If  $\phi \circ f \notin \vec{\text{SP}}(f)$ , then, for some points  $x_1, \dots, x_n \in X$  and vectors  $a_1, \dots, a_n \in A$ , we have

$$\mathbf{1} = \sum_{i=1}^n a_i(\phi \circ f(x_i)\mathbf{1} - f(x_i)).$$

Therefore,

$$1 = \phi(\mathbf{1}) = \sum_{i=1}^n \phi(a_i)(\phi(f(x_i)) - \phi(f(x_i))) = 0.$$

This is a contradiction. Hence  $\phi \circ f \in \vec{\text{SP}}(f)$ .

Conversely, take  $\lambda \in \vec{\text{SP}}(f)$ , and consider the ideal  $I$  in  $A$  generated by

$$S = \{\lambda(x)\mathbf{1} - f(x) : x \in X\}.$$

Since  $\mathbf{1} \notin I$ , the ideal  $I$  is proper. So there is a character  $\phi \in \mathfrak{M}(A)$  with  $I \subset \ker \phi$ . This means that  $\phi(\lambda(x)\mathbf{1} - f(x)) = 0$ , for all  $x \in X$ , whence  $\lambda = \phi \circ f$ .  $\square$

Next, we show that  $\vec{\text{SP}}(f)$ , for  $f \in C(X, A)$ , is a compact subset of  $C(X)$ . To this end, the following lemma is needed. Recall that a family  $\mathcal{F}$  of continuous functions from  $X$  into a metric space  $(Y, d)$  is *equicontinuous* if, for every  $x \in X$  and every  $\varepsilon > 0$ , there is a neighborhood  $U_x$  of  $x \in X$  such that

$$d(f(x), f(y)) < \varepsilon \quad (y \in U_x, f \in \mathcal{F}).$$

**Lemma 3.2.** *If a family  $\mathcal{F}$  of functions in  $C(X, A)$  is equicontinuous, then the family  $\mathcal{G}$  defined by*

$$\mathcal{G} = \{\phi \circ f : \phi \in \mathfrak{M}(A), f \in \mathcal{F}\} \quad (3.1)$$

*is equicontinuous.*

*Proof.* Let  $x_0 \in X$  and  $\varepsilon > 0$ . Since  $\mathcal{F}$  is equicontinuous, there is a neighbourhood  $U_0$  of  $x_0$  such that  $\|f(x) - f(x_0)\| < \varepsilon$ , for all  $f \in \mathcal{F}$  and  $x \in U_0$ . Then

$$|\phi \circ f(x) - \phi \circ f(x_0)| \leq \|f(x) - f(x_0)\| < \varepsilon \quad (\phi \in \mathfrak{M}(A), f \in \mathcal{F}, x \in U_0).$$

Hence  $\mathcal{G}$  is equicontinuous.  $\square$

The Arzelà-Ascoli theorem states that a family  $\mathcal{F} \subset C(X)$  is relatively compact if  $\mathcal{F}$  is equicontinuous and pointwise bounded.

**Theorem 3.3.** *For every  $f \in C(X, A)$ , the spectrum  $\vec{\text{SP}}(f)$  is a compact subset of  $C(X)$ .*

*Proof.* That  $\vec{\text{SP}}(f) \subset C(X)$  follows from Theorem 3.1 and the assumption that  $f : X \rightarrow A$  is continuous. To show that  $\vec{\text{SP}}(f)$  is compact in  $C(X)$ , using Arzelà-Ascoli theorem, we only need to show that  $\vec{\text{SP}}(f)$  is uniformly closed, uniformly bounded and equicontinuous.

We prove that  $\vec{\text{SP}}(f)$  is uniformly closed in  $C(X)$ . Take a function  $\lambda \in C(X)$  and assume that  $\lambda \notin \vec{\text{SP}}(f)$ . Then, there exist  $a_1, \dots, a_m \in A$  and  $x_1, \dots, x_m \in X$  such that

$$\mathbf{1} = \sum_{i=1}^m a_i (\lambda(x_i)\mathbf{1} - f(x_i)). \quad (3.2)$$

To get a contradiction, assume  $\lambda \in \overline{\vec{\text{SP}}(f)}$ . Then, there is a sequence  $\{\phi_n\}$  in  $\mathfrak{M}(A)$  such that  $\phi_n \circ f \rightarrow \lambda$ , uniformly on  $X$ . Take  $\varepsilon = (\|a_1\| + \dots + \|a_m\|)^{-1}$ , where  $a_1, \dots, a_m$  are given by (3.2). There is  $N \in \mathbb{N}$ , such that  $\|\phi_n \circ f - \lambda\|_X < \varepsilon$ , for all  $n \geq N$ . This implies that

$$|\phi_N(f(x_i)) - \lambda(x_i)| < \varepsilon \quad (i = 1, 2, \dots, m).$$

Now, by (3.2), we get

$$\mathbf{1} = \phi_N(\mathbf{1}) = \sum_{i=1}^m \phi_N(a_i) (\lambda(x_i) - \phi_N(f(x_i))).$$



Therefore,

$$1 \leq \sum_{i=1}^m |\phi_N(a_i)(\lambda(x_i) - \phi_N(f(x_i)))| < \sum_{i=1}^m \|a_i\| \varepsilon = 1.$$

This is a contradiction, and thus  $\lambda \notin \overline{\vec{\text{SP}}(f)}$ . We conclude that  $\vec{\text{SP}}(f)$  is uniformly closed in  $C(X)$ . That  $\vec{\text{SP}}(f)$  is uniformly bounded follows from the fact that, for every  $\phi \in \mathfrak{M}(A)$ ,

$$\|\phi \circ f\|_X = \sup\{|\phi(f(x))| : x \in X\} \leq \|\phi\| \sup\{\|f(x)\| : x \in X\} = \|f\|_X.$$

Finally, by taking  $\mathcal{F} = \{f\}$ , Lemma 3.2 implies that  $\vec{\text{SP}}(f)$  is equicontinuous.

The set  $\vec{\text{SP}}(f)$  being uniformly closed, uniformly bounded and equicontinuous is a compact subset of  $C(X)$ .  $\square$

Now, analogous to [3, Proposition 5.17], we prove that the set-valued mapping  $f \mapsto \vec{\text{SP}}(f)$  of  $\mathcal{A}$  into the family of compact sets in  $C(X)$  is upper semi-continuous.

**Lemma 3.4.** *Suppose  $f_n \rightarrow f$  in  $\mathcal{A}$ ,  $\lambda_n \in \vec{\text{SP}}(f_n)$  and  $\lambda_n \rightarrow \lambda$  in  $C(X)$ . Then  $\lambda \in \vec{\text{SP}}(f)$ .*

*Proof.* Towards a contradiction, assume  $\lambda \notin \vec{\text{SP}}(f)$ . Then there exists a finite set of points  $x_1, \dots, x_n$  in  $X$  and vectors  $a_1, \dots, a_n$  in  $A$  such that

$$\mathbf{1} = \sum_{i=1}^n a_i(\lambda(x_i)\mathbf{1} - f(x_i)). \quad (3.3)$$

Take  $\varepsilon = 1/(2 \sum \|a_i\|)$ . By the assumption, there is  $N \in \mathbb{N}$  such that

$$\|f_N - f\|_X < \varepsilon, \quad \|\lambda_N - \lambda\|_X < \varepsilon.$$

Since  $\lambda_N \in \vec{\text{SP}}(f_N)$ , by Theorem 3.1, there is some  $\phi \in \mathfrak{M}(A)$  such that  $\lambda_N = \phi \circ f_N$ . Using (3.3), we have

$$\begin{aligned} 1 &= \left| \sum_{i=1}^n \phi(a_i)(\lambda(x_i) - \phi \circ f(x_i)) \right| \\ &\leq \sum_{i=1}^n \|a_i\| |\lambda(x_i) - \lambda_N(x_i) + \phi \circ f_N(x_i) - \phi \circ f(x_i)| \\ &\leq \sum_{i=1}^n \|a_i\| (\|\lambda - \lambda_N\|_X + \|f_N - f\|_X) \\ &< 2\varepsilon \sum_{i=1}^n \|a_i\| = 1. \end{aligned}$$

This is absurd. Hence  $\lambda \in \vec{\text{SP}}(f)$ .  $\square$

**Theorem 3.5.** *The mapping  $f \mapsto \vec{\text{SP}}(f)$  of  $\mathcal{A}$  into the family of compact sets in  $C(X)$  is upper semi-continuous.*

*Proof.* Suppose the mapping is not upper semi-continuous at some  $f_0 \in \mathcal{A}$ . Hence, there is a neighbourhood  $\Omega$  of  $\vec{sP}(f_0)$  in  $C(X)$  and a sequence  $\{f_n\}$  in  $\mathcal{A}$  such that  $f_n \rightarrow f_0$  and  $\vec{sP}(f_n) \not\subset \Omega$ . Set  $\mathcal{F} = \{f_0, f_1, f_2, \dots\}$  and define  $\mathcal{G}$  as in (3.1). Since  $f_n \rightarrow f_0$  uniformly on  $X$ , the family  $\mathcal{F}$  is equicontinuous on  $X$ . By Lemma 3.2, the family  $\mathcal{G}$  is equicontinuous. Since  $\mathcal{F}$  is uniformly bounded,  $\mathcal{G}$  is uniformly bounded, and, therefore, relatively compact.

Now, consider the mapping  $\vec{sP} : \mathcal{F} \rightarrow 2^{\mathcal{G}}$ ,  $f \mapsto \vec{sP}(f)$ . By Lemma 3.4 and [3, Lemma 5.16], this mapping is upper semi-continuous at  $f_0$ , which contradicts our primary assumption.  $\square$

**The  $\mathfrak{A}$ -valued spectrum.** When  $\mathcal{A}$  is an admissible  $A$ -valued function algebra on  $X$  and  $\mathfrak{A} = \mathcal{A} \cap C(X)$ , then  $\vec{sP}(f) \subset \mathfrak{A}$ , for every  $f \in \mathcal{A}$ . In this case, it is convenient to denote the vector-valued spectrum by  $\vec{sP}_{\mathfrak{A}}(f)$  and call it the  *$\mathfrak{A}$ -valued spectrum*. If the mapping  $\tilde{f} : \mathfrak{M}(A) \rightarrow \mathfrak{A}$ ,  $\phi \mapsto \phi \circ f$ , is continuous (which is the case when  $\mathfrak{A}$  is a uniform algebra) then  $\vec{sP}_{\mathfrak{A}}(f)$  is compact in  $\mathfrak{A}$  since  $\vec{sP}_{\mathfrak{A}}(f) = \tilde{f}(\mathfrak{M}(A))$ . In general, however, it is unknown if  $\tilde{f}$  is continuous or  $\vec{sP}_{\mathfrak{A}}(f)$  is compact in  $\mathfrak{A}$ . We remark that  $\vec{sP}_{\mathfrak{A}}(f)$  is always closed in  $\mathfrak{A}$ .

#### 4. THE $A$ -VALUED SPECTRUM

Let  $\mathcal{A}$  be an admissible Banach  $A$ -valued function algebra on  $X$ . Take  $f \in \mathcal{A}$  and consider the function  $\tilde{f} : \mathfrak{M}(A) \rightarrow \mathfrak{A}$ ,  $\phi \mapsto \phi \circ f$ . By Theorem 3.1, we have

$$\vec{sP}(\tilde{f}) = \{\psi \circ \tilde{f} : \psi \in \mathfrak{M}(\mathfrak{A})\}.$$

Note that every  $\lambda \in \vec{sP}(\tilde{f})$  is a function of  $\mathfrak{M}(A)$  into  $\mathbb{C}$ . Also, every  $a \in A$  induces a function  $\hat{a} : \mathfrak{M}(A) \rightarrow \mathbb{C}$ , the Gelfand transform of  $a$ .

**Definition 4.1.** The  *$A$ -valued spectrum* of a function  $f \in \mathcal{A}$  is defined by

$$\vec{sP}_A(f) = \{a \in A : \hat{a} \in \vec{sP}(\tilde{f})\}. \quad (4.1)$$

**Theorem 4.2.** For every  $f \in \mathcal{A}$ , we have

$$f(X) \subset \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathcal{A})\} \subset \vec{sP}_A(f). \quad (4.2)$$

If every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ , then

$$\vec{sP}_A(f) = \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathcal{A})\}. \quad (4.3)$$

Equality (4.3) is an analogue of (1.1).

*Proof.* The first inclusion in (4.2) is obvious since, for every  $x \in X$ , the evaluation homomorphism  $\mathcal{E}_x : f \mapsto f(x)$  is an  $A$ -character and  $f(X) = \{\mathcal{E}_x(f) : x \in X\}$ .

To prove the second inclusion in (4.2), take an  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$  and let  $a = \Psi(f)$ . Take a finite set  $\phi_1, \dots, \phi_n$  of elements in  $\mathfrak{M}(A)$  and  $g_1, \dots, g_n$  of functions in  $\mathfrak{A}$ , and consider the function

$$g = \sum_{i=1}^n g_i(\hat{a}(\phi_i)\mathbf{1} - \phi_i \circ f).$$

Let  $\psi = \Psi|_{\mathfrak{A}}$ . Then  $\psi \in \mathfrak{M}(\mathfrak{A})$  and thus  $\psi(\mathbf{1}) = 1$ . We have

$$\begin{aligned} \psi(g) &= \psi\left(\sum_{i=1}^n g_i(\hat{a}(\phi_i)\mathbf{1} - \phi_i \circ f)\right) = \sum_{i=1}^n \psi(g_i)(\hat{a}(\phi_i) - \psi(\phi_i \circ f)) \\ &= \sum_{i=1}^n \psi(g_i)(\phi_i(a) - \phi_i(\Psi f)) \\ &= \sum_{i=1}^n \psi(g_i)(\phi_i(a) - \phi_i(a)) = 0. \end{aligned}$$

This implies that  $g \neq \mathbf{1}$ . Hence  $a \in \vec{sP}_A(f)$ , that is,  $\Psi(f) \in \vec{sP}_A(f)$ .

Now, assume that every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ . Take an element  $a \in \vec{sP}_A(f)$ . We show the existence of some  $A$ -character  $\Psi$  such that  $a = \Psi(f)$ . Since  $a \in \vec{sP}_A(f)$ , we have  $\hat{a} \in \vec{sP}(\tilde{f})$ . Hence there is a character  $\psi \in \mathfrak{M}(\mathfrak{A})$  such that  $\hat{a} = \psi \circ \tilde{f}$ . Assume  $\psi$  lifts to the  $A$ -character  $\Psi$  of  $\mathcal{A}$ . Then, for every  $\phi \in \mathfrak{M}(A)$ ,

$$\phi(a) = \hat{a}(\phi) = \psi(\tilde{f}(\phi)) = \psi(\phi \circ f) = \phi(\Psi(f)).$$

Since  $A$  is assumed to be semisimple, we get  $a = \Psi(f)$ .  $\square$

*Remark 4.3.* If every character  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ , then by Theorem 2.4 each  $f \in \mathcal{A}$  extends to a function  $F : \mathfrak{M}(\mathfrak{A}) \rightarrow A$ . In this case,  $\vec{sP}_A(f) = F(\mathfrak{M}(\mathfrak{A}))$  which is compact if  $F$  is continuous.

**Corollary 4.4.** *If  $\mathfrak{A} = C(X) \cap \mathcal{A}$  is a natural function algebra on  $X$ , then  $\vec{sP}_A(f) = f(X)$ , for all  $f \in \mathcal{A}$ . In particular,  $\vec{sP}_A(f)$  is compact in  $A$ .*

*Proof.* Since  $\mathfrak{A}$  is natural, the only characters of  $\mathfrak{A}$  are the point evaluation homomorphisms  $\varepsilon_x$  ( $x \in X$ ). Therefore,  $\mathcal{E}_x$  ( $x \in X$ ) are the only  $A$ -characters of  $\mathcal{A}$ . From (4.3), we get  $\vec{sP}_A(f) = f(X)$  for all  $f \in \mathcal{A}$ .  $\square$

## 5. EXAMPLES

In this section, we identify the  $A$ -valued spectrum in certain algebras.

**Example 5.1.** Let  $\mathcal{A} = C(X, A)$ . Then  $\mathfrak{A} = C(X)$  is natural. By Corollary 4.4, we have  $\vec{sP}_A(f) = f(X)$ , for all  $f \in C(X, A)$ .

**Example 5.2.** Let  $(X, \rho)$  be a compact metric space. An  $A$ -valued *Lipschitz function* is a function  $f : X \rightarrow A$  such that

$$L(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{\rho(x, y)} : x, y \in X, x \neq y \right\} < \infty. \quad (5.1)$$

Denoted by  $\text{Lip}(X, A)$ , the space of  $A$ -valued Lipschitz functions on  $X$  is an  $A$ -valued function algebra on  $X$ , called the  *$A$ -valued Lipschitz algebra*. For  $f \in \text{Lip}(X, A)$ , the Lipschitz norm of  $f$  is defined by  $\|f\|_L = \|f\|_X + L(f)$ . It is easily verified that  $(\text{Lip}(X, A), \|\cdot\|_L)$  is an admissible Banach  $A$ -valued function

algebra and  $\text{Lip}(X) = \text{Lip}(X, A) \cap C(X)$ , where  $\text{Lip}(X) = \text{Lip}(X, \mathbb{C})$  is the classic complex-valued Lipschitz algebra. It is proved in [14] that  $\text{Lip}(X)$  is natural. Therefore, by Corollary 4.4, we have  $\vec{sP}_A(f) = f(X)$ , for all  $f \in \text{Lip}(X, A)$ .

**Example 5.3.** Let  $K$  be a compact set in the complex plane. Let  $P_0(K, A)$  be the algebra of the restriction to  $K$  of all polynomials  $p(z) = a_n z^n + \cdots + a_1 z + a_0$  with coefficients  $a_0, a_1, \dots, a_n$  in  $A$ . Let  $R_0(K, A)$  be the algebra of the restriction to  $K$  of all rational functions of the form  $p(z)/q(z)$ , where  $p(z)$  and  $q(z)$  are polynomials with coefficients in  $A$ , and  $q(\lambda) \in \text{Inv}(A)$ , whenever  $\lambda \in K$ .

The algebras  $P_0(K, A)$  and  $R_0(K, A)$  are  $A$ -valued function algebras on  $K$ , and their uniform closures in  $C(K, A)$ , denoted by  $P(K, A)$  and  $R(K, A)$ , are  $A$ -valued uniform algebras. When  $A = \mathbb{C}$ , we drop  $A$  and write  $P(K)$  and  $R(K)$ , which are complex uniform algebras.

For  $P(K, A)$ , we have  $\mathfrak{A} = P(K)$  and it is known that the character space of  $P(K)$  is naturally identified with  $\hat{K}$  the polynomially convex hull of  $K$  ([8]). Since  $\mathfrak{A}$  is a uniform algebra, by Theorems 2.4 and 2.5, every  $f \in P(K, A)$  extends to a function  $F : \hat{K} \rightarrow A$ . We see that, for all  $f \in P(K, A)$ ,

$$\vec{sP}_A(f) = \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathfrak{A})\} = \{F(\psi) : \psi \in \mathfrak{M}(\mathfrak{A})\} = F(\hat{K})$$

For  $R(K, A)$ , we have  $\mathfrak{A} = R(K)$  and it is known that  $R(K)$  is natural ([8]). By Corollary 4.4, we have

$$\vec{sP}_A(f) = f(K) \quad (f \in R(K, A)).$$

The final example shows that the notion of  $A$ -valued spectrum coincides with the notion of Waelbroeck spectrum [5, Definition 6].

**Example 5.4** (Tensor Products). For a Banach function algebra  $\mathfrak{A}$  on  $X$ , consider the algebraic tensor product  $\mathfrak{A} \otimes A$ . By [3, Theorem 42.6], there is a linear mapping  $T : \mathfrak{A} \otimes A \rightarrow \mathfrak{A}A$  such that

$$T\left(\sum_{i=1}^n f_i \otimes a_i\right) = \sum_{i=1}^n f_i a_i. \quad (5.2)$$

The mapping  $T$  is, in fact, an algebra isomorphism so that the tensor product  $\mathfrak{A} \otimes A$  can be seen as an admissible  $A$ -valued function algebra on  $X$ . The mapping  $T$  in (5.2) extends to an isometric isomorphism of the injective tensor product  $\mathfrak{A} \hat{\otimes}_\epsilon A$  onto the uniform closure  $\overline{\mathfrak{A}A}$  of  $\mathfrak{A}A$  (see [1]). We identify every  $f \in \mathfrak{A} \otimes A$  with its image  $Tf$  given by (5.2). If  $\|\cdot\|_\epsilon$  denote the injective tensor norm, then  $\|f\|_\epsilon = \|f\|_X$ , for all  $f \in \mathfrak{A} \otimes A$ .

In general, let  $\|\cdot\|_\gamma$  be an algebra cross-norm on  $\mathfrak{A} \otimes A$ , and let  $\mathfrak{A} \hat{\otimes}_\gamma A$  be its completion.

**Lemma 5.5.**  $\mathfrak{A} \hat{\otimes}_\gamma A$  is a Banach  $A$ -valued function algebra on  $X$ .

*Proof.* We need to show that every  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$  is a continuous function of  $X$  into  $A$  and  $\|f\|_X \leq \|f\|_\gamma$ . Given  $\varepsilon > 0$ , the element  $f$  has a representation of the form  $f = \sum_{n=1}^\infty f_n$ , with  $f_n \in \mathfrak{A} \otimes A$ , and

$$\|f\|_\gamma \leq \sum_{n=1}^\infty \|f_n\|_\gamma < \|f\|_\gamma + \varepsilon.$$

Since  $\|\cdot\|_\gamma$  is a cross-norm, we have  $\|f_n\|_X = \|f_n\|_\epsilon \leq \|f_n\|_\gamma$ , for all  $n \geq 1$ , and thus the series  $\sum f_n$  converges uniformly, whence  $f$  is a continuous function of  $X$  into  $A$  and  $\|f\|_X \leq \|f\|_\gamma$ .  $\square$

**Lemma 5.6.** *If  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$  and  $\phi \in A^*$  then  $\phi \circ f \in \mathfrak{A}$  and  $\|\phi \circ f\| \leq \|\phi\| \|f\|_\gamma$ . In particular, the algebra  $\mathfrak{A} \widehat{\otimes}_\gamma A$  is admissible.*

*Proof.* First, assume  $f = \sum_{i=1}^n f_i \otimes a_i \in A \otimes \mathfrak{A}$ . Then, for every  $\phi \in A^*$ ,

$$\phi \circ f = \sum_{i=1}^n \phi(a_i) f_i \in \mathfrak{A}.$$

Let  $\mathfrak{A}_1^*$  and  $A_1^*$  denote the closed unit ball of  $\mathfrak{A}^*$  and  $A^*$ , respectively. In case  $\|\phi\| = 1$ , we have

$$\begin{aligned} \|\phi \circ f\| &= \sup\{|\psi(\phi \circ f)| : \psi \in \mathfrak{A}_1^*\} \\ &\leq \sup\{|\psi(\phi \circ f)| : \psi \in \mathfrak{A}_1^*, \phi \in A_1^*\} \\ &= \sup\left\{\left|\sum_{i=1}^n \psi(f_i) \phi(a_i)\right| : \psi \in \mathfrak{A}_1^*, \phi \in A_1^*\right\} \\ &= \|f\|_\epsilon \leq \|f\|_\gamma. \end{aligned} \tag{5.3}$$

In general case, we have  $\|\phi \circ f\| \leq \|\phi\| \|f\|_\gamma$ , for all  $\phi \in A^*$  and  $f \in \mathfrak{A} \otimes A$ .

Next, consider  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$ . There is a sequence  $\{f_n\}$  in  $\mathfrak{A} \otimes A$  that converges to  $f$  with respect to  $\|\cdot\|_\gamma$ . Since  $\|\phi \circ f_n - \phi \circ f_m\| \leq \|\phi\| \|f_n - f_m\|_\gamma$ , for all  $m, n$ , the sequence  $\{\phi \circ f_n\}$  is Cauchy in  $\mathfrak{A}$  whence it converges to some  $h \in \mathfrak{A}$ . Since  $\|f_n - f\|_X \leq \|f_n - f\|_\gamma$ , we have  $f_n(x) \rightarrow f(x)$ , for all  $x \in X$ , and thus

$$h(x) = \lim_{n \rightarrow \infty} \phi(f_n(x)) = \phi(f(x)) = \phi \circ f(x).$$

We see that  $h = \phi \circ f \in \mathfrak{A}$  and

$$\|\phi \circ f\| = \lim_{n \rightarrow \infty} \|\phi \circ f_n\| \leq \|\phi\| \lim_{n \rightarrow \infty} \|f_n\|_\gamma = \|\phi\| \|f\|_\gamma. \quad \square$$

**Lemma 5.7.** *Let  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$  and  $\psi \in \mathfrak{M}(\mathfrak{A})$ .*

- (1) *With respect to the weak\* topology of  $A^*$ , the mapping  $\tilde{f} : A^* \rightarrow \mathfrak{A}$  given by  $\tilde{f}(\phi) = \phi \circ f$ , is continuous on bounded subsets of  $A^*$ .*
- (2) *The mapping  $g : A^* \rightarrow \mathbb{C}$ ,  $\phi \mapsto \psi(\phi \circ f)$  is continuous.*
- (3) *The character  $\psi$  lifts to some  $A$ -character  $\Psi \in \mathfrak{M}_A(\mathcal{A})$ .*

*Proof.* (1) First, assume  $f = \sum_{i=1}^n f_i \otimes a_i \in \mathfrak{A} \otimes A$ . Given  $\phi_0 \in A^*$  and  $\varepsilon > 0$ , define a neighborhood  $U_0$  of  $\phi_0$  as follows:

$$U_0 = \left\{ \phi \in A^* : |\phi(a_i) - \phi_0(a_i)| < \frac{\varepsilon}{\sum_{i=1}^n \|f_i\|}, 1 \leq i \leq n \right\}.$$

Then, for each  $\phi \in U_0$ ,

$$\|\tilde{f}(\phi) - \tilde{f}(\phi_0)\| = \|\phi \circ f - \phi_0 \circ f\| \leq \sum_{i=1}^n \|f_i\| |\phi(a_i) - \phi_0(a_i)| < \varepsilon.$$

We see that, in this case,  $\tilde{f}$  is continuous on  $A^*$ .

In general case where  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$ , take, for every  $\varepsilon > 0$ , an element  $f_0 \in \mathfrak{A} \otimes A$  such that  $\|f - f_0\|_\gamma < \varepsilon$ . Assume  $\{\phi_\alpha\}$  is a bounded net in  $A^*$  that converges to  $\phi_0$ , in the weak\* topology. Suppose  $\|\phi_\alpha\| \leq M$ , for all  $\alpha$ . Since  $\tilde{f}_0 : A^* \rightarrow \mathfrak{A}$  is continuous, there is  $\alpha_0$  such that  $\|\tilde{f}_0(\phi_\alpha) - \tilde{f}_0(\phi_0)\| < \varepsilon$ , for  $\alpha \geq \alpha_0$ . Now, for  $\alpha \geq \alpha_0$ , we have

$$\begin{aligned} \|\tilde{f}(\phi_\alpha) - \tilde{f}(\phi_0)\| &\leq \|\phi_\alpha \circ f - \phi_\alpha \circ f_0\| + \|\phi_\alpha \circ f_0 - \phi_0 \circ f_0\| + \|\phi_0 \circ f_0 - \phi_0 \circ f\| \\ &\leq \|\phi_\alpha\| \|f - f_0\|_\gamma + \varepsilon + \|\phi_0\| \|f - f_0\|_\gamma \\ &< (M + \|\phi_0\| + 1)\varepsilon. \end{aligned}$$

This implies that  $\tilde{f}$  is continuous on bounded subsets of  $A^*$ .

(2) follows from Corollary 3.11.4 in [11] and the fact that  $g$  is continuous on bounded subsets of  $A^*$ . (3) follows from (2) and Theorem 2.4.  $\square$

That every  $\psi \in \mathfrak{M}(\mathfrak{A})$  lifts to some  $\Psi \in \mathfrak{M}_A(\mathfrak{A} \widehat{\otimes}_\gamma A)$  can be verified explicitly as follows. First, define a mapping

$$\Psi_0 : \mathfrak{A} \otimes A \rightarrow A, \quad \Psi_0 \left( \sum_{i=1}^n f_i \otimes a_i \right) = \sum_{i=1}^n \psi(f_i) a_i.$$

The mapping  $\Psi_0$  is an algebra homomorphism with  $\phi(\Psi_0(f)) = \psi(\phi \circ f)$ , for all  $\phi \in A^*$ . It is easily verified that  $\|\Psi_0(f)\| \leq \|f\|_\gamma$ , for all  $f \in \mathfrak{A} \otimes A$ . Being a continuous homomorphism on a dense subspace of  $\mathfrak{A} \widehat{\otimes}_\gamma A$ ,  $\Psi_0$  can extend to a homomorphism  $\Psi : \mathfrak{A} \widehat{\otimes}_\gamma A \rightarrow A$  with the desired properties.

We summarize the above discussion in the following statement.

**Theorem 5.8.** *The algebra  $\mathfrak{A} \widehat{\otimes}_\gamma A$  is an admissible Banach  $A$ -valued function algebra on  $X$  that satisfies all conditions in Theorem 2.4.*

In [5], the Waelbroeck spectrum of elements  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$  has been studied. An examination of the results shows that, given  $\psi \in \mathfrak{M}(\mathfrak{A})$ , the mapping  $\psi \otimes I_A$  there coincides with the  $A$ -character  $\Psi$  to which  $\psi$  lifts. Hence, the Waelbroeck spectrum  $\sigma_W(f)$  of every  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$  [5, Definition 6] coincides with the  $A$ -valued spectrum  $\vec{\text{SP}}_A(f)$ . That is

$$\sigma_W(f) = \vec{\text{SP}}_A(f) = \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathfrak{A} \widehat{\otimes}_\gamma A)\}.$$

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